*Game Theory* **Mid-Term Exam**
*Each problem is worth 10 points.* Due: November 11

1. Provide two references for your project and write an abstract of the contents of the two papers in 100 words or less.

2. Consider the two person zero-sum matrix game with the player who picks the row ($U$, $C$ or $D$) being the maximizer and the player who picks the column ($L$, $M$ or $R$) being the minimizer.

| | L | M | R |
|---|---|---|---|
| U | 1 | 1 | 3 |
| C | 0 | 3 | 0 |
| D | 3 | 0 | 0 |

The above table indicates the payoffs to Player 1 (the row player).

(i) Determine the security levels of the players (in pure strategies). Does the game admit a saddle point in pure strategies?

(ii) Compute the saddle-point value of the game in mixed strategies and obtain a pair of mixed strategies for the players that achieve this value. Is this a unique saddle point solution?

3. Consider the non-zero sum game

| | L | R |
|---|---|---|
| U | -10,5 | 2,-2 |
| D | 1,-1 | -1,1 |

Obtain all mixed-strategy noncooperative Nash equilibrium solutions to this game.

4. Consider the two-person zero-sum game on the unit square:

$$L(u, v) = (u + x)(v + x); \qquad 0 \le u \le 1, \quad 0 \le v \le 1,$$

where $u$ is the minimizer and $v$ is the maximizer. The value of the game is determined by *nature*, i.e., it depends on a random variable $x$, which takes on a value $+1$ with probability $p$ and a value $-1$ with probability $1 - p$, where $0 < p < 1$.

(i) Assume that the players do not have access to the outcome of nature's pick, but know the probability $p$. Obtain the upper and lower values of the game (as a function of $p$). Does the game admit a saddle point in pure strategies?

(ii) Now assume that the players do have access to the outcome of nature's pick? What would the upper and lower values be in this case?

5. (i) Consider a firm which is interested in developing a new product called the widget. Let us suppose that the cost of developing the widget is $c$ and the benefit from developing the widget is $p^2$ where $p$ is a uniformly distributed in $[0, 1]$. Thus, $p$ can be viewed as the type of the firm. Nature picks $p$ and reveals it to the firm and then the firm has to decide whether to develop the widget or not. For what values of $p$ will the firm decide to develop the widget?

(ii) Now consider a consortium of two firms who are both interested developing the widget. Under the rules of the consortium if one firm develops the widget, then the knowledge should be shared with the other firm. As before, the cost of developing the widget is $c$. Let $p_i$ be the type of firm $i$ and let the benefit to firm $i$ be $p_i^2$. In other words, if firm 1 decides to develop the widget and firm 2 decide not to develop, then firm 1's payoff is $p_1^2 - c$ while firm 2's payoff is $p_2^2$ since it benefits from the widget without having to pay for its development. Assume that the $p_i$'s are independent and are uniformly distributed in $[0, 1]$. Each firm must independently decide whether to develop the widget or not. Find a Bayesian Nash equilibrium for this game.

6. Find the trembling-hand perfect Nash equilibrium in the following two-person game:

| | L | R |
|---|---|---|
| U | 1,1 | 0,-3 |
| D | -3,0 | 0,0 |

7. Give an example of a mixed strategy in strategic or normal form which is not a behavioral strategy for the extensive form of the game.

8. For the signalling example considered in the last lecture on Oct. 30, show that the following are perfect Bayesian equilibria:

$$\{(L, L), (u, d), p = .5, q \leq 2/3\}$$

and

$$\{(R, L), (u, u), p = 0, q = 1\}\,.$$

Also, show that there are no perfect Bayesian Nash equilibria with the sender playing $(R, R)$ or $(L, R)$.

9. Consider the following two-person noncooperative game:

| | L | R |
|---|---|---|
| U | 5,1 | 0,0 |
| D | 4,4 | 1,5 |

(i) Find all the pure-strategy and mixed-strategy Nash equilibria of this game.

(ii) Using an appropriate equilibrium concept, find an equilibrium which gives a higher payoff to both players compared to the mixed-strategy equilibrium computed in part (i).

10. In this problem, we will demonstrate the Harsanyi purification theorem. Consider the two-person noncooperative game

| | L | R |
|---|---|---|
| U | 0,0 | 0,-1 |
| D | 1,0 | -1,3 |

(i) Find a mixed-strategy Nash equilibrium for the above game.

(ii) Now, consider the following incomplete information game where $\epsilon \in (0, 1)$ and $\theta_1$ and $\theta_2$ are the types of the row player (Player 1) and the column player (Player

| | L | R |
|---|---|---|
| U | $\epsilon\theta_1, \epsilon\theta_2$ | $\epsilon\theta_1, -1$ |
| D | $1, \epsilon\theta_2$ | $-1, 3$ |

2), respectively. Suppose that $\theta_i$ are independent random variables and are uniformly distributed in $[0, 1]$. Find a Bayesian equilibrium for this problem and show that, in the limit as $\epsilon \to 0$, this equilibrium tends to the mixed-strategy Nash equilibrium obtained in part (i).

**The next question is a bonus question. It will not used in determining the grading curve for the class, but it can be used to boost an individual student's grade after the grading curve has been determined.**

11. **(Bonus question)** I gave an outline of the proof of Theorem 2, Theorem 6 and the discussion after Theorem 6 regarding bilinear forms of $\phi_i$ in Rosen's paper. Please read Rosen's paper and write down the precise proofs of the two theorems and the discussion following Theorem 6 in Section 3 of the paper. If you don't understand any terminology in the paper, please find books in the library to help you understand it. In short, treat this as a very short project! The result in reference [4] in the paper regarding the conditions for the stability of a matrix can also be found in many books on state-space control theory (books that are used in ECE415 and GE323).