Singular Control with State Constraints

Amarjit Budhiraja

Department of Statistics & Operations Research

University of North Carolina at Chapel Hill

with R. Atar and R.J. Williams

Outline.

1. Problem Description. State space, Controlled dynamics, Admissible controls, Cost criterion.
2. Nonnegative Cost, Unbounded Domain.
   - Setup.
   - HJB Equation: Variational inequality with gradient constraints. Constrained viscosity solutions.
   - Main Result: Unique solvability by the value function.
   - Sketch of Proof.
3. General Cost, Bounded Domain.
   - Setup.
   - HJB equation and the main uniqueness result.
   - Remarks on assumptions and the “no arbitrage" condition.
4. Unresolved issues.

Problem Description.

- State Space: $\mathbb{X}$ Closed, convex subset of $\mathbb{R}^d$.
- Uncontrolled Process: $x + B_t$,

  $x \in \mathbb{X}$, B a (b, Σ) Brownian motion. $b \in \mathbb{R}^d$, Σ a $d \times d$ n .n.d. matrix.
- Controlled Dynamics: $X_t = x + B_t + GY_t$,

  G is a $d \times p$ matrix, $d \leq p$.
- Control: Y - a p dimensional RCLL process, loc. b.v. paths, adapted (to some underlying filtration $\{\mathcal{F}_t\}$).

Problem Description Contd.

Admissible Control:

1. Y has increments in the closed convex cone $\mathbb{Y}$:

$$Y(0), Y(t) - Y(s) \in \mathbb{Y},\ 0 \leq s \leq t < \infty.$$

   (E.g. $\mathbb{Y} = \mathbb{R}^p_+$.)

2. State Constraints:

$$X_t = x + B_t + GY_t \in \mathbb{X},\ \text{for all } t \geq 0,\ a.s.$$

Let $\mathcal{A}(x)$ denote the family of all admissible controls.

General Description Contd.

Cost Criterion:

$$J(x,Y) = \mathbb{E}\int_{[0,\infty)} e^{-\beta s}(\ell(X_s)ds + h \cdot dY_s),$$

$\ell : \mathbb{X} \to I\!R$, $h \in I\!R^p$.

Goal: Characterize the value function

$$V(x) = \inf_{Y \in \mathcal{A}(x)} J(x,Y)$$

as the (unique) solution of a suitable PDE.

- Nonnegative Cost: $\ell(x) \geq 0, h\cdot \mathbb{Y} \subset I\!R_+$, $\mathbb{X}$ is a cone.
- General cost, bounded $\mathbb{X}$.

Nonnegative Cost.

$\ell(x) \geq 0, h\cdot \mathbb{Y} \subset \mathbb{R}_+$, $\mathbb{X}$ is a cone.

Dynamic Programming Heuristic. Recall: $X_t = x + B_t + GY_t$.

One Candidate control: Apply no control for $0 \leq s \leq \delta$ and then control optimally.

$$\begin{aligned} V(x) &\leq \mathbb{E}_x\left(\int_0^\delta e^{-\beta s}\ell(X_s)ds + e^{-\beta\delta}V(X_\delta)\right) \\ &= \mathbb{E}_x\left(\int_0^\delta e^{-\beta s}\ell(X_s)ds + e^{-\beta\delta}V(x) - e^{-\beta\delta}\int_0^\delta \mathcal{L}V(X_s)ds\right) \end{aligned}$$

where $\mathcal{L} \doteq -(1/2)\mathrm{Tr}(\Sigma D^2) - b\cdot D$. Dividing by $\delta \to 0$:
$(\mathcal{L} + \beta)V(x) - \ell(x) \leq 0.$

Equality holds if it is "optimal" to apply no control around x.

Dynamic Programming Heuristic (Contd.)

Another control: Jump by $\delta y \in \mathbb{Y}$ at time 0 & then go optimally.

$$V(x) \leq \delta h \cdot y + V(x + \delta G y), \; y \in \mathbb{Y}.$$

$$-DV(x) \cdot \frac{Gy}{||Gy||} - h \cdot \frac{y}{||Gy||} \leq 0.$$

Thus defining $\mathcal{H}(p) \doteq \sup_{y \in \mathbb{Y}: |Gy|=1} \{-(Gy \cdot p + y \cdot h)\}$:

$$\mathcal{H}(DV(x)) \leq 0.$$

Equality holds if it is optimal to apply control at x.

HJB equation.

Both $\mathcal{H}(DV(x))$ and $(\mathcal{L}+\beta)V(x) - \ell(x)$ are at most 0 and one of them is exactly 0.

$$\underbrace{((\mathcal{L}+\beta)\psi(x) - \ell(x)) \vee \mathcal{H}(D\psi(x))}_{F(x,\psi(x),D\psi(x),D^2\psi(x))} = 0$$

Boundary Condition: Fix $x \in \partial\mathbb{X}$.

Suppose First: $\mathcal{H}(DV(x)) = 0$.

Recall: $\mathcal{H}(DV(x)) \doteq \sup_{y\in\mathbb{Y}:|Gy|=1}\{-(Gy\cdot DV(x) + y\cdot h)\}$.

Suppose that the supremum is attained at y^*:

HJB (contd.)

$$0 = \mathcal{H}(DV(x)) = -(Gy^* \cdot DV(x) + y^* \cdot h).$$

Must have: $Gy^*.n(x) \leq 0$ where $n(x)$ is the outward normal at x.

Let $\phi \in C^2(\mathbb{X})$ be such that $V - \phi$ is minimized at x. Then:

$$D\phi(x) = DV(x) + \theta n(x),$$

for some $\theta \geq 0$. Thus:

$$\mathcal{H}(D\phi(x)) = \mathcal{H}(DV(x) + \theta n(x)) \geq -\theta G y^* \cdot n(x) \geq 0.$$

So $F(x, V(x), D\phi(x), D^2\phi(x)) \geq 0$

– Can argue similarly if $\mathcal{H}(DV(x)) < 0$.

BC: V is a supersolution on $\mathbb{X}$. (Soner)

Viscosity Solutions (Lions, Crandall, Ishii, ...)

$-$ In general no C^2 solutions- a weaker notion of a solution is needed.
Basic idea: Suppose u is a classical subsolution, i.e.,

$$u \text{ is } C^2 \text{ and } \; F(x, u(x), Du(x), D^2u(x)) \leq 0.$$

Then if ϕ is C^2 and $u - \phi$ has a maximum at x then

$$Du(x) = D\phi(x), \;\; D^2u(x) \leq D^2\phi(x).$$

$$\begin{aligned} F(x, u(x), D\phi(x), D^2\phi(x)) &= F(x, u(x), Du(x), D^2\phi(x)) \\ &\leq F(x, u(x), Du(x), D^2u(x)) \leq 0. \end{aligned}$$

Thus: $F(x, u(x), D\phi(x), D^2\phi(x)) \leq 0.$

Definition (Constrained Viscosity Solution).

$$F(x, \psi(x), D\psi(x), D^2\psi(x)) = 0. \qquad (1)$$

(1) For $S \subset \mathbb{X}$, a function $\psi : \overline{S} \to [0, \infty)$ is a supersolution of (1) on S if $\forall\, x \in S$, $\varphi \in C^2(S)$ s.t. $\psi - \varphi$ has a global minimum at x:

$$F(x, \psi(x), (D\varphi)(x), (D^2\varphi)(x)) \geq 0.$$

(2) For $S \subset \mathbb{X}$, a function $\psi : \overline{S} \to [0, \infty)$ is a subsolution of (1) on S if $\forall\, x \in S$, $\varphi \in C^2(S)$ s.t. $\psi - \varphi$ has a global maximum at x:

$$F(x, \psi(x), (D\varphi)(x), (D^2\varphi)(x)) \leq 0.$$

(3) ψ is said to be a constrained viscosity solution of (1) on $\mathbb{X}$ if it is a subsolution of (1) on $\mathbb{X}^o$ and a supersolution of (1) on $\mathbb{X}$.

Some Amusing Examples.

$\mathbb{X} = \mathbb{Y} = I\!R_+, G = 1, \beta = 1, h = 0$

(a) Suppose $\ell(x) \equiv 1$, $b = 0, \Sigma = 1$

$$V(x) = \inf_Y J(x, Y) = \inf_Y \mathbb{E} \int_0^\infty e^{-t} dt = 1.$$

$$F(\cdots) = (-\frac{1}{2}\psi''(x) + \psi(x) - 1) \vee (-D\psi(x)).$$

Then $\psi_c(x) \equiv c$ is a solution for every $c \in [0, 1]$.

Examples (Contd.).

$\mathbb{X} = \mathbb{Y} = I\!R_+, G = 1, \beta = 1, h = 0$

(b) Suppose $\ell(x) = x + 1, b = -1, \Sigma = 0.$

$$\begin{aligned} V(x) &= \inf_Y \mathbb{E} \int_0^\infty e^{-t}(X_t + 1)dt \\ &= \int_0^x e^{-t}(x - t + 1)dt + \int_x^\infty e^{-t} 1 dt = x + e^{-x}. \end{aligned}$$

$$F(\cdots) = (\psi'(x) + \psi(x) - x - 1) \vee (-D\psi(x)).$$

V is a solution,
$\psi(x) = \max(x + ce^{-x}, C)$, $c \in [0, 1]$, $C \in (c, 1]$ is a solution as well.

- $\psi(x) = x$ solves $F(\cdots) = 0$ on $[0, \infty)$ but does'nt satisfy the boundary condition: Subdifferential at 0 is the set $(-\infty, 1]$.

Standing Assumptions

- Controllability: $\mathbb{X} \cap G\mathbb{Y} \neq \emptyset$.
- On the cones $\mathbb{X}$ and $\mathbb{Y}$:
  - Union of $\mathbb{X}$ and $G\mathbb{Y}$ is (strictly) contained in a half space: $\exists$ unit vector $e_1 \in I\!R^d$, $a_1 > 0$ s.t. $\forall\ x, y$ in $\mathbb{X}$, $\mathbb{Y}$
    $x \cdot e_1 \geq a_1 |x|, \qquad Gy \cdot e_1 \geq a_1 |Gy|.$
  - $\mathbb{Y}$ is (strictly) contained in a half space: $\exists$ unit vector $e_2 \in I\!R^p$ and $a_2 > 0$ s.t. $\forall\ y \in \mathbb{Y}$, $y \cdot e_2 \geq a_2 |y|$.
- On the running cost ℓ:
  - $\exists\ m_\ell \in \mathbb{N}_0$ such that $\forall\ x \in \mathbb{X}$,

$$c_1 |x|^{m_\ell} - c_2 \leq \ell(x) \leq c_3(|x|^{m_\ell} + 1).$$

  - $\sup_{|x-y|<\delta} \frac{|\ell(x)-\ell(y)|}{1+|x|^{m_\ell}+|y|^{m_\ell}} \to 0$ as $\delta \to 0$.

Main Result (*Ann. Prob.* (2006)).

Let $C_{\mathrm{pol},+}(\mathbb{X})$ be the class of continuous nonnegative functions with (at most) polynomial growth.

$$C^{\mathrm{c}}_{\mathrm{pol},+}(\mathbb{X}) \doteq \{f \in C_{\mathrm{pol},+}(\mathbb{X}) | f \text{ has compact sub-level sets}\}.$$

$$C_{b,+}(\mathbb{X}) \doteq \{f \in C_{\mathrm{pol},+}(\mathbb{X}) | f \text{ is bounded}\}.$$

Theorem.

(1) Recall: $c_{\ell,1}|x|^{m_\ell} - c_{\ell,2} \leq \ell(x) \leq c_{\ell,3}(|x|^{m_\ell} + 1)$.

– If $m_\ell > 0$ then V is in $C^{\mathrm{c}}_{\mathrm{pol},+}(\mathbb{X})$.
– If $m_\ell = 0$ then V is in $C_{b,+}(\mathbb{X})$.

(2) Solvability. V is a constrained viscosity solution of (1) on $\mathbb{X}$.

Main Result (Contd.).

(3) Uniqueness. Let Assumption 1 hold.

Assumption 1. Either Σ is non-degenerate or $e_1 \cdot b > 0$.

- If $m_\ell > 0$, V is the only such solution in the class $C^{\mathrm{c}}_{\mathrm{pol},+}(\mathbb{X})$.
- If $m_\ell = 0$, V is the maximal solution in the class $C_{\mathrm{pol},+}(\mathbb{X})$.

Suppose, in addition, Assumption 2 holds.

Assumption 2. For some $c_h > 0$, $h \cdot y \geq c_h |Gy|, \quad y \in \mathbb{Y}$.

Then uniqueness holds in $C_{\mathrm{pol},+}(\mathbb{X})$, $m_\ell \geq 0$.

Remark.

One can construct (one dimensional) examples to show:

- If Assumption 1 is dropped, in the case $m_\ell > 0$: Multiple solutions may exist (even) in $C^{\mathrm{c}}_{\mathrm{pol},+}(\mathbb{X})$.
- If Assumption 2 is dropped:
  - In the case $m_\ell > 0$: Uniqueness in $C^{\mathrm{c}}_{\mathrm{pol},+}(\mathbb{X})$ but multiple solutions may exist in $C_{\mathrm{pol},+}(\mathbb{X})$.
  - In the case $m_\ell = 0$: multiple solutions may exist in $C_{b,+}(\mathbb{X})$ (value function is the maximal one).

Sketch for Uniqueness.

For $r > 0$, denote

$$\mathbb{X}_r \doteq \{x \in \mathbb{X} : x \cdot e_1 < r\}, \qquad \partial_r \doteq \{x \in \mathbb{X} : x \cdot e_1 = r\}.$$

For $Y \in \mathcal{A}(x)$ and $X \doteq x + B + GY$, set

$$\sigma \equiv \sigma_r \doteq \inf\{t : X_t \notin \mathbb{X}_r\}.$$

Let ϕ be any function in $C_+(\partial_r)$. Define

$$J_{r,\phi}(x, Y) \doteq \mathbb{E}[\int_{[0,\sigma]} e^{-\beta s}[\ell(X_s)ds + h \cdot dY_s] + e^{-\beta\sigma}\phi(X_\sigma)],$$

Let

$$V_{r,\phi}(x) \doteq \inf_{Y \in \mathcal{A}(x)} J_{r,\phi}(x, Y), \quad x \in \overline{\mathbb{X}}_r.$$

Sketch(Contd.).

Theorem 1. Let $u, v \in C_+(\overline{\mathbb{X}}_r)$ be two constrained viscosity solutions of (1) on $\mathbb{X}_r$ with the Dirichlet boundary condition ϕ on ∂_r. Then $u = v$.

$-$ Comparison Principle: If u is a subsolution and v is a supersolution then $u \leq v$ (Maximum principle for $u - v$).

$-$ A typical argument: If $F(x, r, p, A)$ is r- coercive then maximum principle holds.

(r-coercive: $F(x, r_1, p, A) \leq F(x, r_2, p, A)$ implies $r_1 \leq r_2$.)

Sketch(Contd.).

Eg. if u and v are C^2, u subsoln., v supersoln. and $x \in \mathbb{X}^o$ is such that $u - v$ is maximized at x, then:

$$Du(x) = Dv(x) \text{ and } D^2u(x) \leq D^2v(x).$$

Thus

$$\begin{aligned} F(x, u(x), Du(x), D^2u(x)) &\leq 0 \\ &\leq F(x, v(x), Dv(x), D^2v(x)) \\ &\leq F(x, v(x), Du(x), D^2u(x)). \end{aligned}$$

Coercivity gives $u(x) \leq v(x)$.

– Here coercivity of F fails... A careful construction of suitable strict subsolution is needed.

Sketch(Contd.).

Theorem 2. $V_{r,\phi}$ is a constrained viscosity solution of (1) on $\mathbb{X}_r$ with the Dirichlet boundary condition ϕ on ∂_r.

Proof. DPP...

Theorem 3. Let $u \in C_+(\mathbb{X})$ be a constrained viscosity solution of (1) on $\mathbb{X}$. Then for $x \in \overline{\mathbb{X}}_r$

$$u(x) = \inf \mathbb{E}[\int_{[0,\sigma_r]} e^{-\beta s}[\ell(X_s)ds + h \cdot dY_s] + e^{-\beta\sigma_r} u(X_{\sigma_r})]. \quad (V)$$

– Let $r \to \infty$ in (V).

General cost, $\mathbb{X}$ Bounded.

$-$ Generalized Brownian networks of Harrison and Williams.
Controlled Dynamics:

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + GY_t$$

$-$ b, σ Lip. Control cone $\mathbb{Y}$ and admissible controls as before.
Cost Criterion:

$$J(x, Y) = \mathbb{E}\int_{[0,\infty)} e^{-\beta s}(\ell(X_s)ds + h \cdot dY_s),$$

ℓ continuous , $\ell, h \cdot y$ may take negative values.
Goal: Characterize the value function $V(x) = \inf_{Y \in \mathcal{A}(x)} J(x, Y)$ in terms of the associated HJB equation.

Standing Assumptions.

- Controllability: $G\mathbb{Y} = \mathbb{R}^d$.
- Control cone Geometry: $\exists\ e \in \mathbb{R}^p$ such that

$$\inf_{y \in \mathbb{Y}:|Gy|=1} e\cdot y > 0.$$

- Cost bounded from below:

$$\sup_{x \in \mathbb{X}} \sup_{Y \in \mathcal{A}(x)} \mathbb{E} \int_0^\infty e^{-\beta s}(h\cdot Y_s)^- ds < \infty. \qquad (A1)$$

- Integration by parts holds: $\forall\ Y \in \mathcal{A}(x)$, $x \in \mathbb{X}$,

$$\beta \int_0^\infty e^{-\beta s} h\cdot Y_s ds = \int_{[0,\infty)} e^{-\beta s} d(h\cdot Y_s). \qquad (A2)$$

Value function Properties.

Proposition. $V(x) \in (-\infty, \infty)$ for each $x \in \mathbb{X}$. V is Lipschitz on $\mathbb{X}$.

Sketch.

- $V(x) > -\infty$ follows from (A1).
- Upper bound, $x \in \mathbb{X}^o$: Let B be a small enough ball such that $x \in B \subset \mathbb{X}^o$. Let Control be the normal reflection on the boundary. Easy to check associated cost is finite.
- Lipschitz property: State can be moved from any $x_1 \in \mathbb{X}$ to any other $x_2 \in \mathbb{X}$, instantaneously, by exercising a control and paying a cost proportional to $|x_1 - x_2|$.

HJB equation.

Let $\mathcal{L}f = -\frac{1}{2}\text{Tr}(\sigma D^2 f \sigma') - b\cdot Df$
Recall $\mathcal{H}(q) = \sup_{y\in\mathbb{Y}:|Gy|=1}\{-Gy\cdot q - h\cdot y\}$.
HJB equation:

$$\underbrace{((\mathcal{L}+\beta)\psi(x) - \ell(x)) \vee \mathcal{H}(D\psi(x))}_{F(x,\psi(x),D\psi(x),D^2\psi(x))} = 0 \qquad (2)$$

- Constrained Viscosity solution: Subsolution on $\mathbb{X}^o$ and supersolution on $\mathbb{X}$.
- Assumption (A3): $\inf_{q\in\mathbb{R}^p}\mathcal{H}(q) < 0$
- Remark: If assumption fails then every continuous function is a supersolution and if ψ is a subsolution then so is $\psi - c$ for $c > 0$.

Characterization.

Theorem. Under the standing assumptions:

- Solvability: V is a constrained viscosity solution of (2).
- Uniqueness: Assumption (A3) is necessary and sufficient for V to be only constrained viscosity solution in $C_b(\mathbb{X})$.

No Arbitrage Condition (Harrison and Williams):

$$\{y \in \mathbb{Y} : Gy = 0 \text{ and } h \cdot y \leq 0\} = \{0\}. \qquad (A4)$$

Theorem. No arbitrage condition (A4) implies (A1), (A2), (A3).

Sketch of (A4) implies (A3).

Recall

$$\inf_{q\in I\!\!R^p} \mathcal{H}(q) = \inf_{q\in I\!\!R^p} \sup_{y\in\mathbb{Y}:|Gy|=1} \{-Gy\cdot q - h\cdot y\} < 0 \qquad (A3)$$

$$\{y \in \mathbb{Y} : Gy = 0 \text{ and } h\cdot y \leq 0\} = \{0\}. \qquad (A4)$$

- For $\widetilde{h} \in I\!\!R^p$, let

$$\mathcal{E}(\widetilde{h}) = \{y \in \mathbb{Y} : Gy = 0, \widetilde{h}\cdot y \leq 0\}.$$

  Note: $\mathcal{E}(\widetilde{h}) \supset \{0\}$.

- Also: $\{h\} \subset \{\widetilde{h} \in I\!\!R^p | \mathcal{E}(\widetilde{h}) = \{0\}\} \doteq H$.
- H is open. Let ε be s.t. $\mathcal{E}(h^\varepsilon) = \{0\}$, $h^\varepsilon = h - \varepsilon e_1$.

Sketch (Contd.)

- V^ε be the value function with $h \mapsto h^\varepsilon$.
- $V^\varepsilon(x) \in (-\infty, \infty)$ and V^ε solves the HJB equation (for h^ε).
- Construct a C^2 function ϕ such that for some $x_0 \in \mathbb{X}^o$, $V^\varepsilon(x_0) - \phi(x_0)$ is maximized at x_0.
- Using the subsolution property of V^ε one can show that $\mathcal{H}(q_0) < 0$ with $q_0 = D\phi(x_0)$.

Other Issues.

- Free boundary problem: Optimal process a reflected diffusion.
- Smoothness of the free boundary.
- Interior regularity of Value function. For Singular Control (without state constraints): Evans, Ishii and Koike, Shreve and Soner, Menaldi and Robin.
- Ergodic Control. (without state constraints: Karatzas, Taksar, Kurtz and Stockbridge, Robin and Menaldi).
- Numerical Methods.