Reflected Brownian Motion (RBM) in a Quadrant: Tail Behavior of the Stationary Distribution

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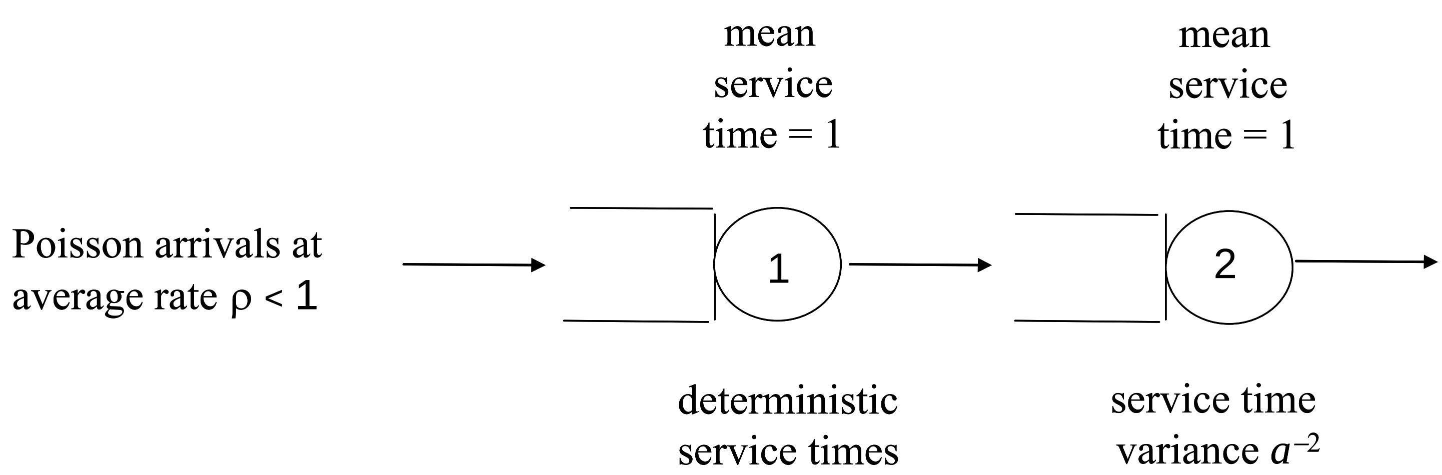
Abstract. We consider a two-dimensional Brownian motion confined to the non-negative quadrant by oblique reflection at the boundary. (In this talk, attention is restricted to a specially structured example, but the theory generalizes.) Analytic methods are used to derive a simple formula for the asymptotic decay rate of the stationary distribution along each axis, and an interpretation in terms of minimum-energy paths is provided.

* joint work with J. J. Hasenbein, L. A. Shepp and S. R. S. Varadhan

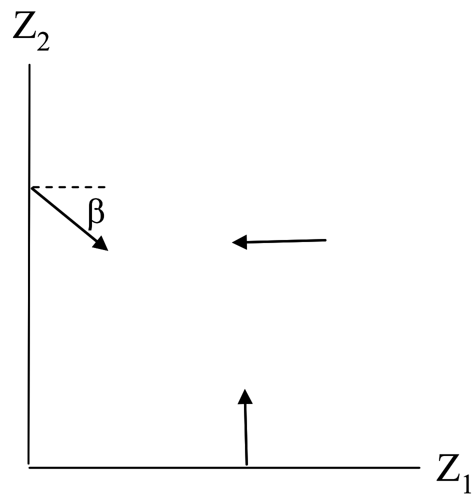
Outline of the Talk

- Description of the specially structured example
- Analytic approach to asymptotic decay rates
- Large deviations approach to asymptotic decay rates
- A taste of the general theory for RBM in a quadrant

A Simple Queueing Network



Its Approximating RBM



$Z_1(t) = X_1(t) \qquad\qquad + Y_2(t)$

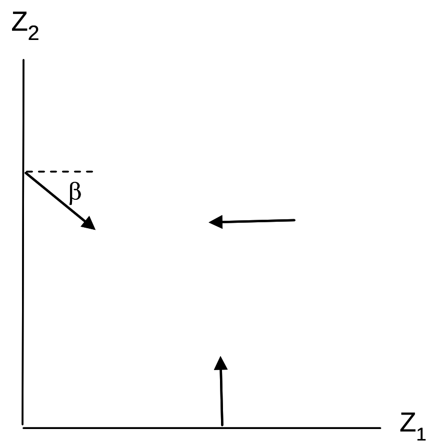
$Z_2(t) = X_2(t) + Y_1(t) - aY_2(t)$

$\tan \beta = a$

drift of X is $(-1,0)$

covariance of X is I

Stationary Distributions and Associated Transforms



$$\mu(A) = \lim_{t\to\infty} \frac{1}{t} E\{\int_0^t 1_A(Z(s))\, ds\}$$

$$\varphi(\lambda) = \int exp(\lambda\cdot z)\, \mu(dz) \quad \text{for } \lambda \in \Re^2$$

$$\mu_i(A) = \lim_{t\to\infty} \frac{1}{t} E\{\int_0^t 1_A(Z_i(s))\, dY_i(s)\} \qquad i = 1, 2.$$

$$\varphi_i(\lambda_i) = \int exp(\lambda_i z_i)\, \mu_i(dz_i) \quad \text{for } \lambda_i \in \Re, \qquad i = 1, 2.$$

The stationary distribution μ has a density function $p(z)$, and each stationary pushing measure μ_i has a density $p_i(z_i)$. Moreover,

$$p_1(z_1) = \tfrac{1}{2}\, p(z_1,0) \quad \text{and} \quad p_2(z_2) = \tfrac{1}{2}\, p(0, z_2)\ .$$

Our Goal is to Determine

$$sup\ \{\lambda_1 > 0: \varphi_1(\lambda_1) < \infty\}$$

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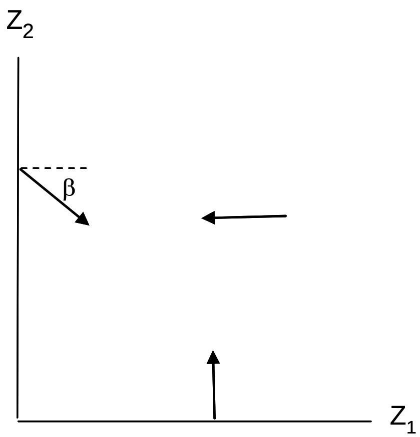
A Bit of History

In 1978 Varadhan derived an integral representation for the transform $\varphi_2(\lambda_2)$. In 1983 that result was generalized by Marjorie Foddy in her unpublished Ph.D. dissertation (Stanford Mathematics). In November of 2005, starting from the integral representation of $\varphi_2(\lambda_2)$, Shepp concluded that

$$sup\ \{\lambda_2 > 0\colon \varphi_2(\lambda_2) < \infty\} = \frac{2a}{1+a^2}.$$

Analytic Approach to Asymptotic Decay Rates

The Basic Adjoint Relationship (BAR)



$$\int Af(z)\,\mu(dz) + \sum_{i=1}^{2} \int D_i f(z_i)\,\mu_i(dz_i) = 0$$

for suitable test functions f

Conjectural Tranform Identities

Let

$$Q(\lambda) = \frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2^2 - \lambda_1 \quad \text{for } \lambda \in \Re^2$$

Putting the test function

$$f(z) = exp(\lambda \cdot z)$$

into the (BAR) and blithely assuming all terms are finite, we get the following:

$$Q(\lambda)\varphi(\lambda) + \lambda_2\varphi_1(\lambda_1) + (\lambda_1 - a\lambda_2)\varphi_2(\lambda_2) = 0.$$

In particular,

$$\lambda_2 \varphi_1(\lambda_1) + (\lambda_1 - a\lambda_2)\varphi_2(\lambda_2) = 0 \quad \text{if} \quad Q(\lambda) = 0.$$

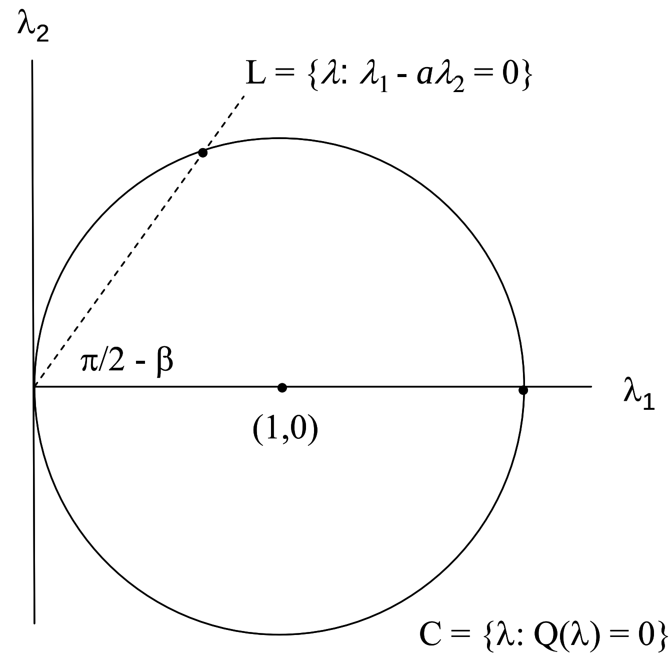
If true, this means that

(1) $$\varphi_1(\lambda_1) = \frac{(a\lambda_2 - \lambda_1)\varphi_2(\lambda_2)}{\lambda_2}$$

and

(2) $$\varphi_2(\lambda_2) = \frac{\lambda_2 \varphi_1(\lambda_1)}{(a\lambda_2 - \lambda_1)}$$

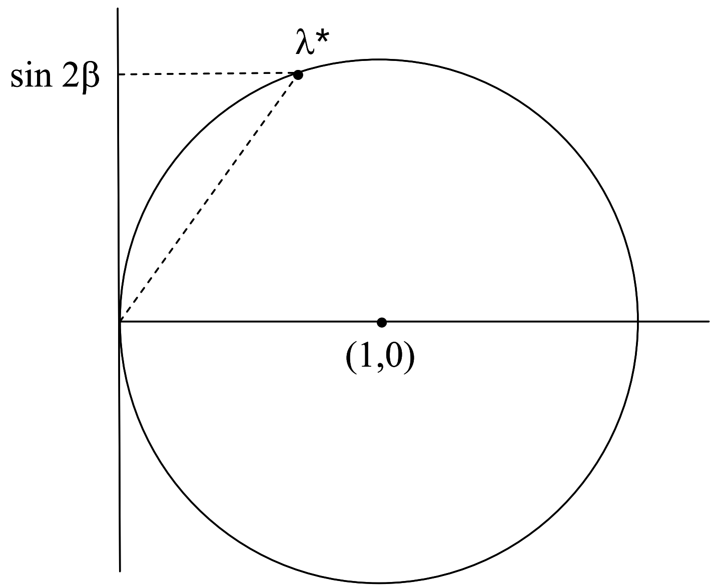
on the circle C pictured at right.



A Critical Point

The point λ* at which L and C intersect has

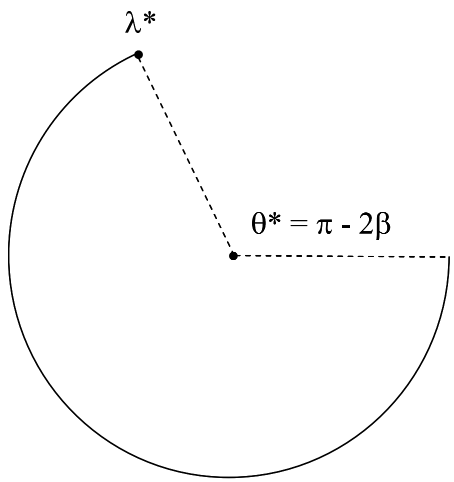
$$\lambda_2^* = \frac{2a}{1+a^2} = \sin 2\beta.$$



Defining the Set C*

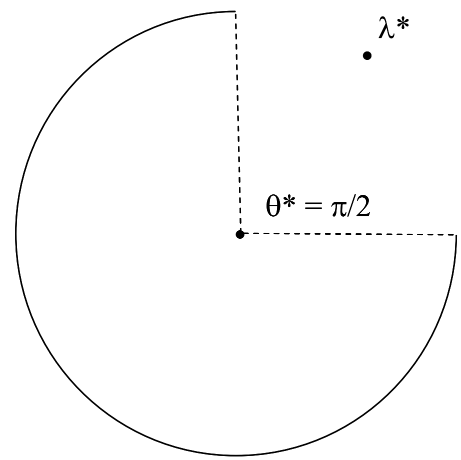
Create C* from C by deleting the arc specified below (endpoints of the arc are also deleted).

Case 1



$a \leq 1$
$\beta \leq \pi/4$

Case 2



$a > 1$
$\beta > \pi/4$

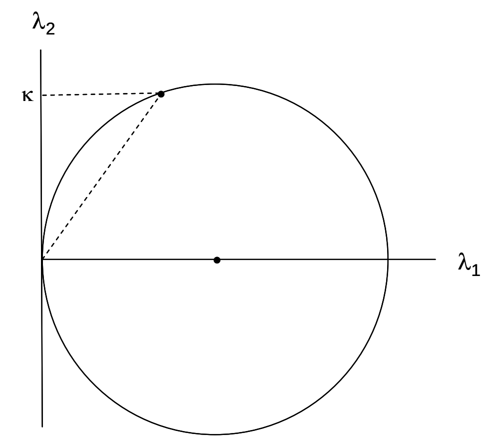
The Main Analytic Results

Theorem. For each $\lambda \in C^*$, both $\varphi_1(\lambda_1)$ and $\varphi_2(\lambda_2)$ are finite, and

(3) $$\lambda_2 \varphi_1(\lambda_1) + (\lambda_1 - a\lambda_2)\varphi_2(\lambda_2) = 0 \; .$$

Corollary. $sup\ \{\lambda_1 > 0\colon \varphi_1(\lambda_1) < \infty\} = 2$ and $sup\ \{\lambda_2 > 0\colon \varphi_2(\lambda_2) < \infty\} = \kappa$

where $\kappa = \begin{cases} \sin 2\beta & if \quad 0 < \beta \le \pi/4 \\ 1 & if \;\; \pi/4 < \beta < \pi/2 \end{cases}$



A Key Step in Proving the Theorem

Consider a point λ on the lower half of the circle C, so that $\lambda_2 < 0$ and $\lambda_1 - a\lambda_2 > 0$. From Ito's formula we obtain the following:

$$exp(\lambda\cdot Z(t)) - 1 = \lambda_2 \int_0^t exp(\lambda_1 Z_1(s))\, d\,Y_1(s) + (\lambda_1 - a\lambda_2)\int_0^t exp(\lambda_2 Z_2(s))\, d\,Y_2(s) + M_\lambda(t),$$

Rearranging terms, taking expectations and dividing by t gives

$$|\lambda_2|\, \varphi_1^t(\lambda_2) + \frac{1}{t} E[exp(\lambda\cdot Z(t)) - 1] = (\lambda_1 - a\lambda_2)\varphi_2^t(\lambda_2) < \infty.$$

Letting $t \to \infty$ and using Fatou's lemma, we have

$$|\lambda_2|\, \varphi_1(\lambda_1) \le (\lambda_1 - a\lambda_2)\varphi_2(\lambda_2) < \infty\ .$$

Proof of the Corollary

The transform identity (3) can be stated equivalently as follows:

$$\varphi_2(u) = H(u)\,\varphi_2(-u), \qquad 0 \le u < \kappa$$

$$\varphi_1(u) = K(u)\,\varphi_2(-G(u)), \quad 0 \le u < 2$$

where

$$H(u) = \frac{au + F(u)}{au - F(u)} \qquad \text{and} \qquad F(u) = 1 - \sqrt{1-u^2}\,,$$

$$K(u) = \frac{u + a\,G(u)}{G(u)} \qquad \text{and} \qquad G(u) = \sqrt{2u - u^2}\ .$$

It follows from these formulas that

- $\varphi_1(\lambda_1) \to \infty$ as $\lambda_1 \uparrow 2$

- $\varphi_2(\lambda_2) \to \infty$ as $\lambda_2 \uparrow \kappa$ if $a \le 1$

- $\varphi_2(\lambda_2) \to \mathrm{c} > 0$ as $\lambda_2 \uparrow 1$ if $a > 1$

  $\Rightarrow \int_0^\infty e^x \mu_2(dx) < \infty$ if $a > 1$

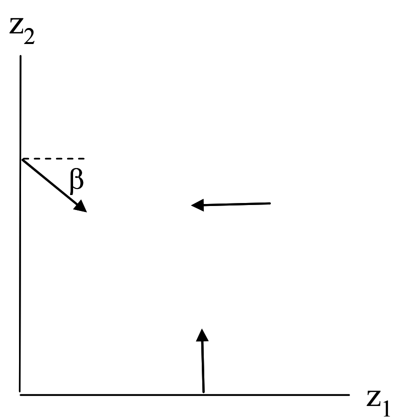
- $\frac{d}{dz}\varphi_2(\lambda_2) \to \infty$ as $\lambda_2 \uparrow 1$ if $a > 1$

  $\Rightarrow \int_0^\infty x e^x \mu_2(dx) = \infty$ if $a > 1$

Large Deviations Approach to Asymptotic Decay Rates

A Variational Problem

Let $\mathcal{X}$ be the set of all absolutely continuous paths x: $[0,\infty) \to \mathcal{R}^2$ with $x(0) = 0$. We write $(y,z) = \psi(x)$ to mean that y is the boundary pushing path associated with x, and z is the reflected path that results. Thus $y(\cdot)$ is non-decreasing and $z(\cdot) \geq 0$.



$$z_1(t) = x_1(t) \qquad + \; y_2(t)$$

$$z_2(t) = x_2(t) + y_1(t) - a\, y_2(t)$$

$$\tan \beta = a$$

Given a target point v in the positive quadrant, let $\mathcal{X}(v)$ be the set of paths $x \in \mathcal{X}$ that further satisfy

$$z(T) = v \;\text{ for some } T > 0,$$

For $x \in \mathcal{X}(v)$ we define the energy functional

$$I_v(x) = \frac{1}{2} \int_0^T |\, \dot{x}(t) + (1,0) |^2 \, dt \,.$$

The following *variational problem* arises in large deviations theory for the RBM under study: find

(4) $$I_v^* = inf \; \{ I_v(x) \colon x \in \mathcal{X}(v) \} \,.$$

If x achieves the infimum in (4) and $(y,z) = \psi(x)$, then (x,y,z) is called an *optimal triple* for the target point v, or the *minimum-energy* triple for v.

Avram, Dai and Hasenbein (2001) solved this variational problem for arbitrary v, but their solution is amenable to further reduction, or further simplification.

Large Deviations Heuristic

Recall the density relationships

(5) $$p_1(z_1) = \frac{1}{2}\, p(z_1,0) \quad \text{and} \quad p_2(z_2) = \frac{1}{2}\, p(0, z_2)\,.$$

For each point v in the non-negative quadrant, there is good reason to believe that

(6) $$-\frac{1}{x}\log p(xv) \to I_v^* \quad \text{as} \quad x \to \infty,$$

where I_v^* is the minimum achievable energy in the variational problem above. Of course, (5) and (6) together suggest that

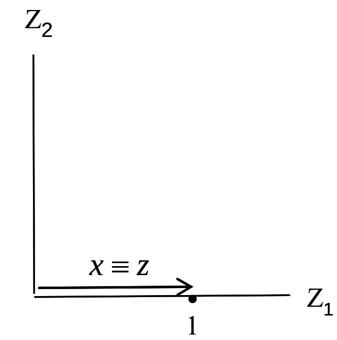
$$-\frac{1}{x}\log p_1(x) \to I_{(1,0)}^* \quad \text{as} \quad x \to \infty, \quad \text{and}$$

$$-\frac{1}{x}\log p_2(x) \to I_{(0,1)}^* \quad \text{as} \quad x \to \infty.$$

Solutions for Two Variational Problems

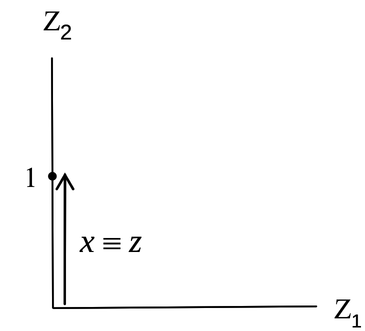
- For $v = (1,0)$ the optimal triple (x,y,z) has $y_1 = y_2 \equiv 0$, and hence $z = x$ (that is, a *direct path* is optimal), with minimum energy

$$I^*_{(1,0)} = 2.$$



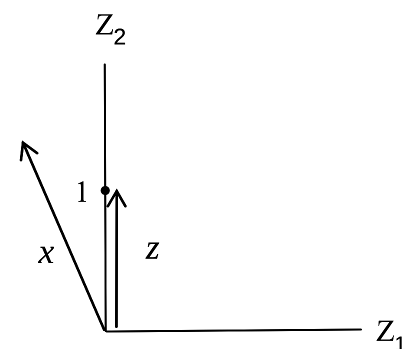
- If $a > 1$ then for $v = (0,1)$ the optimal triple (x,y,z) has $y_1 = y_2 \equiv 0$, and hence $z = x$ (that is, a *direct path* is again optimal), with minimum energy

$$I^*_{(0,1)} = 1.$$



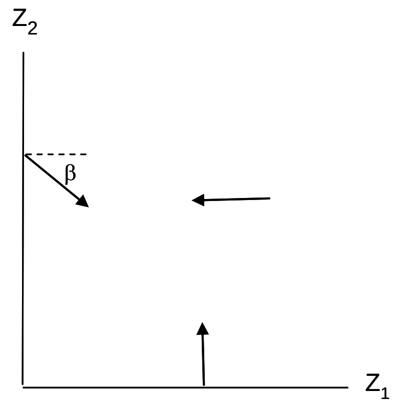
- If $a \leq 1$ then for $v = (0,1)$ the optimal triple (x,y,z) has $y_1 = z_1 \equiv 0$, $\dot{y}_2 > 0$ and $\dot{z}_2 > 0$ (that is, a *bouncy path along the vertical axis* is optimal), with minimum energy

$$I^*_{(0,1)} = \sin 2\beta.$$



A Taste of the General Theory for RBM in a Quadrant

Adding Correlation to Previous Example



Now X has drift vector (-1,0), and covariance matrix

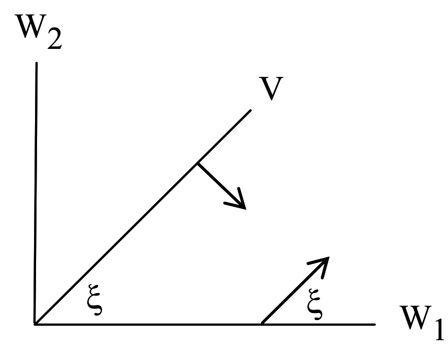
$$\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \quad \text{where} \quad 0 < \rho < 1.$$

Also, assume for simplicity that $a = \rho$, where $a = \tan\beta$ as before.

Let ξ be the angle satisfying $\cos\xi = \rho$, and define a new process W via

$$W(t) = M\,Z(t), \qquad \text{where} \qquad M = \left(\frac{1}{\sin\xi}\right)\begin{bmatrix} 1 & \cos\xi \\ 0 & \sin\xi \end{bmatrix}.$$

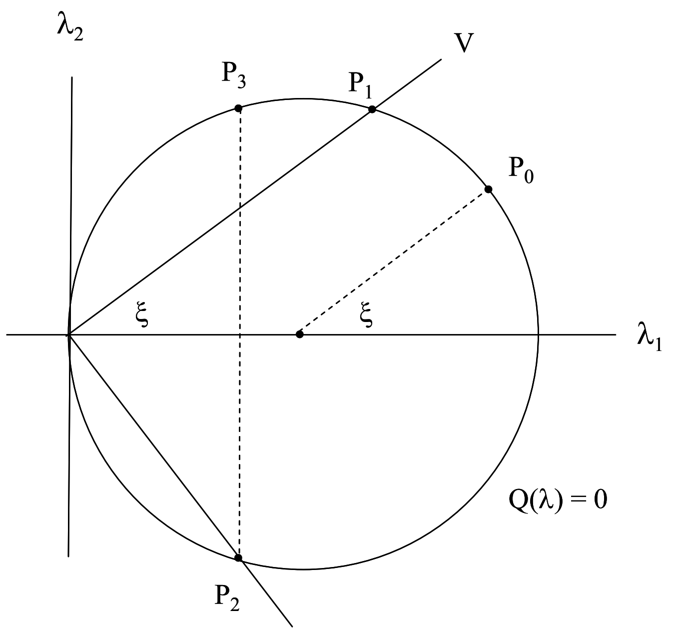
The new process W lives in the wedge pictured below (V denotes the upper boundary ray), with the boundary directions shown. Its covariance matrix is I, and we can take its drift vector to be (-1,0).



Again let

$$Q(\lambda) = \frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2^2 - \lambda_1 \quad .$$

Finding Minimum Energy Path to v = (cos ξ, sin ξ)



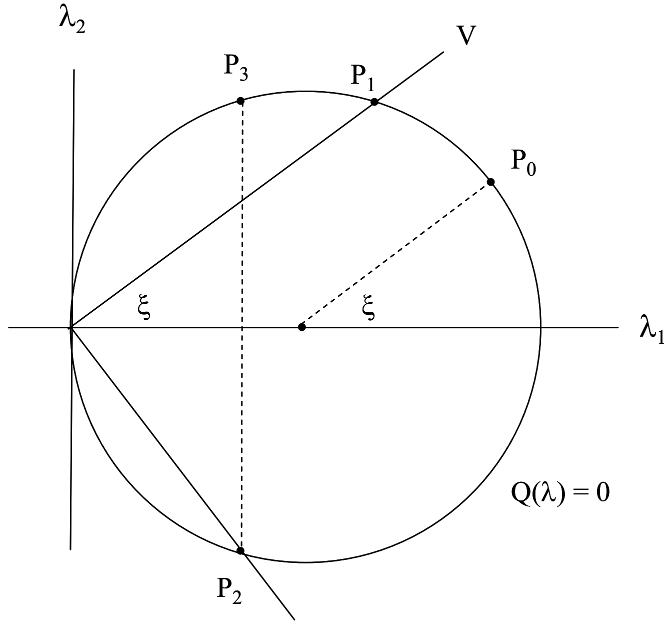
P_0 is the point on the circle having the longest projection onto the upper boundary ray V

P_1 is the point where V intersects the circle

P_2 is the point where the line orthogonal to V intersects the circle

P_3 is the other point on the circle having the same λ_1 value as P_2.

- Energy of direct path to v = length of projection of P_0 onto V, which is $1 + \cos \xi$. Direct path is not optimal, regardless of ξ.



- If $\pi/4 \leq \xi \leq \pi/2$, then a bouncy path along the upper axis is optimal, the minimum achievable energy being

$$I_v^* = \text{ length of the line from origin to } P_1$$

$$= 2 \cos \xi$$

- If $0 < \xi < \pi/4$, then the optimal triple (x,y,z) is a *broken path* of the following kind: z bounces along the *lower* boundary initially, and then takes a direct path to v through the interior of the wedge. The minimum achievable energy is

$$I_v^* = \text{ length of projection of } P_3 \text{ onto V} = \cos \xi + \cos (\pi - 3\xi)$$

References

F. Avram, J. G. Dai and J. J. Hasenbein (2001), Explicit solutions for variational problems in the quadrant, *Queueing Systems*, 37, 259-289.

R. J. Williams (1995), Semimartingale reflecting Brownian motions in the orthant, in *Stochastic Networks*, eds. F. P. Kelly and R. J. Williams, the IMA Volumes in Mathematics and its Applications, Vol. 71 (Springer, New York)

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