Stability and product-form invariant distributions for multidimensional diffusions with state-dependent reflections

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Outline

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- Jump diffusions with state dependent reflections
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Motivation

Identifying stationary distributions of reflected diffusions with jumps-queues with "noisy servers"

Develop a framework for network stability

Motivation (contd.)

Multi-dimensional reflected processes, in particular multi-dimensional reflected diffusions with jumps, arise in a wide variety of applications such as:

- Finance
- Queueing networks in heavy traffic
- Risk theory
- Subsidy-surplus models

Background

- Reiman (Math. Oper. Res., 1984) showed an open queueing network converges weakly to an SRBM (special case) in heavy traffic. Recently, Gamarnik et al. (Annals App. Prob., 2006) established the validity of such heavy traffic steady state approximation.
- Chen and Whitt (Queueing Syst. Theory App., 1993) showed in heavy traffic the number of customers in an open queueing network subject to service interruptions can be approximated by a reflected Brownian motion with jumps in the positive orthant.
- Since then, weak convergence of more complex network models to reflected diffusions with jumps has been established, allowing for example state-dependency (Kushner, *Heavy Traffic Analysis of Controlled Queueing and Communication Networks*, Springer, 2001).
- Wide variety of limit processes: Noah Effect (heavy tails), Joseph Effect (strong dependence) (Whitt, *Stochastic Process Limits*, Springer, 2002).

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A Special Case: SRBM

Semi-martingale reflected Brownian motion (SRBM):

$$t \geq 0, \quad X_t = Y_t + RZ_t$$

where:

- X: $\mathbb{R}^n_+$-valued, continuous semi-martingale.
- Y: $\mathbb{R}^n$-valued Brownian motion with covariance matrix Γ and drift vector Θ. ($Y_0 = x \in \mathbb{R}^n_+$.)
- R: $n \times n$ completely-S real matrix (i.e., $\forall \tilde{R}$ principal submatrix of R, $\exists x \geq 0$ s.t. $\tilde{R}x > 0$).
- Z: $\mathbb{R}^n_+$-valued, continuous process with each Z^i non-decreasing, null at zero and such that $\int_{\mathbb{R}_+} X^i_s dZ^i_s = 0$.

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A Special Case: SRBM (2)

- By writing $X^i = U^i + R_{ii}Z^i$, an (implicit) characterization of the regulator process Z^i of X^i at level 0 is given by:

$$Z_t^i = \frac{1}{R_{ii}} \sup_{s\in[0,t]} \max\{-U_s^i, 0\}$$

- Z also results from applying the multi-dimensional reflection map to the unconstrained or "free" process Y: Z is the minimal element in the space of continuous, null at zero and (componentwise) non-decreasing processes s.t. $Y + RZ \geq 0$.
- SRBM has been studied quite extensively not only because of its pure mathematical significance, but also because of its applied relevance in heavy-traffic limits for stochastic networks.

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SRBM: Related Results

- Reiman and Williams (Probab. Theory Related Fields, 1988) showed that the completely-S property of R is necessary for the existence of SRBM on the orthant.
- Taylor and Williams (Probab. Theory Related Fields, 1993) established sufficiency and uniqueness.
- Reiman and Williams (Probab. Theory Related Fields, 1988) also showed the following boundary property: The regulator processes do not charge the set of times spent in the intersection of two or more faces, i.e., $\forall \mathbf{J} \subset \{1, \ldots, n\}$, $|\mathbf{J}| \geq 2$, and any k in $\mathbf{J}$, a.s.:

$$\int_0^\infty \mathbf{1}\{X_s^j = 0, j \in \mathbf{J}\} dZ_s^k = 0$$

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SRBM: Related Results (2)

- This boundary property is important in characterizing the stationary regime and its corresponding stationary distribution.
- Harrison and Williams (Stochastics, 1987) showed the necessary and sufficient conditions for product-form stationary distribution (marginals independent in stationary regime) are:

$$R^{-1}\Theta < 0 \ \ (for \ \ stationary \ \ regime)$$

$$\Gamma_{ij} = \frac{R_{ji}}{2R_{ii}}\Gamma_{ii} + \frac{R_{ij}}{2R_{jj}}\Gamma_{jj} \ \ (for \ \ separability)$$

($R = (I - P^T)D$, $P \geq 0$ sub-stochastic, transient, $D > 0$ diagonal).

- More recently, Shen et al. (Queueing Syst. Theory App., 2002) also used this boundary property to develop numerical methods for computing the stationary distribution of queueing networks in the heavy traffic limit (SRBM in a hypercube).

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Reflected Diffusions with Jumps: 1-d

- Mazumdar and Guillemin (App. Math. Optim., 1996) studied one-dimensional reflected diffusions, allowing for time and space dependent drift and diffusion coefficients and jumps, i.e.:

$$t \geq 0, \;\; dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \int_0^\infty \mu(dt, dz) + dZ_t$$

- There they related the regulator process Z to the corresponding semi-martingale local time at level 0, $L(t, 0)$, as:

$$Z_t = -\int_0^t \mathbf{1}\{X_s = 0\} b(s, 0) ds + \frac{1}{2} L(t, 0)$$

- Then they used this, and other supporting results, to study the stationary distributions of such processes.

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Semi-martingale Local Times

The *Local Time* of a *càdlàg* semi-martingale X at level $r \in \mathbb{R}$ measures the amount of time that X spends in the neighborhood of r:

$$t \geq 0, \quad L(t,r) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}\{r \leq X_s \leq r + \epsilon\} d[X,X]_s^c \qquad \Big(\sum_{0<s\leq t} |\Delta X_s| < \infty \Big)$$

Time occupation density interpretation (relative to the random "clock" $d[X,X]_s^c$): $\forall f$ bounded Borel measurable we have a.s.:

$$\int_0^t f(X_{s-}) d[X,X]_s^c = \int_{-\infty}^{+\infty} f(r) L(t,r) dr$$

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Multi-dimensional Case

We consider a reflected diffusion with positive and negative jumps, constrained to lie in the positive orthant of $\mathbb{R}^n$:

$$X_t = X_0 + \int_0^t b(s, X_{s-})ds + \int_0^t \gamma(s, X_{s-})dW_s + \sum_{0<s\leq t} \Delta X_s + RZ_t$$

where:

- X is an $\mathbb{R}^n_+$-valued, *càdlàg* semi-martingale such that $\sum_{0<s\leq t} |\Delta X_s| < \infty$ a.s. for each $t > 0$.
- b and γ are Borel functions from $\mathbb{R}_+ \times \mathbb{R}^n_+$ into $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively. We set $a \equiv \gamma\gamma*$.
- W is an n-dimensional, standard Brownian motion.

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Multi-dimensional Case (2)

and:

- $R \in \mathbb{R}^{n \times n}$ is a completely-S matrix with the additional property that every principal submatrix extracted from it is non-singular. Note if R is not diagonal then the reflections are oblique.
- Z^i's are the regulator processes associated to each X^i, i.e., by writing $X^i = U^i + R_{ii} Z^i$, then:

$$Z_t^i = \frac{1}{R_{ii}} \sup_{s \in [0,t]} \max\{-U_s^i, 0\}$$

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Main Results

- If $\exists i \in K \subseteq \{1, \ldots, n\}$ such that $a_{ii}(s, 0_K) > 0$ for a.e. $s \in [0, t]$, then:

$$a.s.: \quad m\{s \in [0,t] : X_s^j = 0, \forall j \in K\} = 0$$

  and:

$$a.e. \;\; s \in [0,t]: \quad P_s^X\{x \in \mathbb{R}_+^n : x_k = 0, \forall k \in K\} = 0$$

- The second result establishes that, for a.e. $s \in [0, t]$, the law of X_s (denoted as P_s^X) does not charge the set $\bigcap_{k \in K}\{x \in \mathbb{R}_+^n : x_k = 0\}$.

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Main Results (2)

- If $\exists i, j \in K \subseteq \{1, \ldots, n\}, i \neq j$, such that $a_{jj}(s, 0_{K\setminus\{i\}}) > 0$ and $a_{ii}(s, 0_{K\setminus\{j\}}) > 0$ for a.e. $s \in [0, t]$, then $\forall q \in K$ we have a.s.:

$$\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\} dZ_s^q = 0$$

- This result extends, under the additional invertibility requirement on principal submatrices of R, the boundary property for SRBMs in the orthant, where there are no jumps and the drift, as well as the diffusion matrix, are constant. It allows for the following pure local time characterization of Z.

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Main Results (3)

- If $\forall i \in \{1,\dots,n\}$ we have $a_{ii}(t,0_{\{i\}}) > 0$ for a.e. $t \in \mathbb{R}_+$, then $\forall i \in \{1,\dots,n\}$ we have a.s.:

$$Z^i_\cdot = \frac{1}{2R_{ii}} L_i(\cdot, 0)$$

- This is an important result for characterizing the stationary distribution of X, as we will see shortly.
- As a corollary of these results, and under the same assumptions as above, we have a.s.:

$$suppt\{dZ^i_s\} = suppt\{L_i(ds,0)\} \subseteq \{s \in \mathbb{R}_+ : \; X^i_s = 0, X^j_s > 0, \forall j \neq i\}$$

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Hyper-rectangular State Spaces

- All the previous results are readily adapted to the case of an hyper-rectangular state space: $\times_{i=1}^{n}[0, u_i]$ with each $u_i > 0$.
- Main motivation: queueing theory, when work arrives to finite queues.
- The model is:

$$X_t = X_0 + \int_0^t b(s, X_{s-})ds + \int_0^t \gamma(s, X_{s-})dW_s + \sum_{0<s\leq t} \Delta X_s + RZ_t - \tilde{R}\tilde{Z}_t$$

Hyper-rectangular State Spaces (2)

- By writing $X^i = V^i - \tilde{R}_{ii}\tilde{Z}^i$, the regulator process of X^i at level u_i can be (implicitly) characterized as:

$$\tilde{Z}^i_t = \frac{1}{\tilde{R}_{ii}} \sup_{s \in [0,t]} \max\{V^i_s - c_i, 0\}$$

- As mentioned, all the previous results are readily adapted to this case. For example, if $\forall i \in \{1, \ldots, n\}$ we have $a_{ii}(t, c_{\{i\}}) > 0$ for a.e. $t \in \mathbb{R}_+$, then $\forall i \in \{1, \ldots, n\}$ we have a.s.:

$$\tilde{Z}^i_\cdot = \frac{1}{2\tilde{R}_{ii}} L_i(\cdot, c_i -)$$

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Product-form Distributions

- There has been much interest in the particular case when the stationary distribution of this class of reflected processes factorizes as the product of its one-dimensional marginals, termed as product-form case (like in the context of classical Markovian queueing networks; see for example the book by Kelly, *Reversibility and Stochastic Networks*, Wiley, 1979).
- Harrison and Williams (Stochastics, 1985) provided necessary and sufficient conditions for a product-form stationary distribution in the case of SRBMs in the orthant.
- Product-form stationary distributions for reflected Lévy processes (including jumps) have been exploited in Bardhan (Stochastic Processes and App., 1995) for example.

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Product-form Distributions (2)

- Negative results for queueing models with Lévy inputs are given in Kella (J. App. Probab., 2000) for a 2-d fluid network or, more recently, for an n-dim Lévy stochastic network in Konstantopoulos et al. (Queueing Syst., 2004).

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State-dependent Drift and Diffusion

We study the case of product-form stationary distributions for a similar reflected diffusion with jumps as before:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \Delta X_t + RdZ_t$$

where, in addition:

- $b_i(x) = b_i(x_i)$, $a_{ij}(x) = a_{ij}(x_i, x_j)$, $a_{ii}(0) > 0$, and $a_{ii}(0+)$ exists and is finite.
- The jump measure driving the jumps in X admits a Markovian predictable compensator of the form $\lambda(\omega)K(X_{t-}(\omega), dz)dt$, where the probability transition kernel $K(x, dz)$ has a separable structure $\prod_{i=1}^{n} K_i(x_i, dz_i)$, with each one-dimensional marginal distribution $K_i(x_i, \cdot)$ such that $K_i(x_i, \{0\}) = K_i(x_i, (-\infty, -x_i)) = 0$ for each $x_i \geq 0$, and the intensity λ is ind. of $(X_{t-})_{t\geq 0}$ and takes the form $\sum_{i=1}^{n} \lambda_i$, with λ_i the jump intensity associated with X^i and the λ_i's ind.

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Main Result

- The stationary density p is in product-form, i.e., $p(x) = \prod_{i=1}^{n} p_i(x_i)$ for a.e. $x \in \mathbb{R}_+^n$ with p_i its i-th one-dimensional marginal, if and only if for each $i, j \in \{1, \ldots, n\}$, $i \neq j$, we have:

$$a_{ij}(x_i, x_j) = \frac{R_{ji}}{2R_{ii}} \frac{a_{ii}(0+)p_i(0+)}{p_i(x_i)} \int_{x_i}^{\infty} p_i(\xi)d\xi \quad + \quad (i \leftrightarrow j)$$

  for P_{ij}-a.e. $(x_i, x_j) \in \mathbb{R}_+^2$, where P_{ij} denotes the joint (stationary) law of the tuple (X_t^i, X_t^j) in $\mathbb{R}_+^2$.

- **Remark.** The following assumptions have been made:
  - A stationary version exists. Moreover, the stationary law of X_t in $\mathbb{R}_+^n$ is absolutely continuous (with density p) w.r.t. Lebesgue measure.
  - Each one-dimensional marginal is such that $p_i(0+)$ exists and is finite.

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Some Remarks

- In the case when there are no jumps and the drift vector $b(\cdot)$ is constant, the product-form condition becomes:

$$a_{ij}(x_i, x_j) = \frac{R_{ji}}{2R_{ii}} a_{ii}(x_i) + \frac{R_{ij}}{2R_{jj}} a_{jj}(x_j) \quad \forall i \neq j$$

- This generalizes the socalled *skew-symmetric* condition for SRBMs.
- Finally, it is easy to see that if the diffusion matrix a has constant off-diagonal elements, then a product-form stationary density is not possible unless: Either the jumps in each coordinate are identically null and the drift b, as well as the diagonal elements in a, are constant; Or a and the reflection matrix R are diagonal (this generalizes the negative result in Konstantopoulos, Last and Lin (Queueing Syst., 2004), shown for n-dimensional reflected Lévy processes).

Some Explicit Examples

Constant Drift, No Jumps. Assuming strictly positive and continuously differentiable diagonal elements in a, one can show that:

$$p(x) = \prod_{i=1}^{n} \frac{a_{ii}(0)p_i(0)}{a_{ii}(x_i)} \exp\left\{-a_{ii}(0)p_i(0) \int_0^{x_i} \frac{d\xi}{a_{ii}(\xi)}\right\}$$

where $p_i(0) = -2\frac{R_{ii}}{a_{ii}(0)}(R^{-1}b)_i$. The separability condition in this case is the same as the one in the previous slide.

- This generalizes the exponential product-form density obtained in the case of SRBM.

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Some Explicit Examples (2)

Normal Reflections, No Jumps. In this case one can show that, under the same assumptions as in the previous slide:

$$p(x) = \prod_{i=1}^{n} \frac{a_{ii}(0)p_i(0)}{a_{ii}(x_i)} \exp\left\{2\int_0^{x_i} \frac{b_i(\xi)}{a_{ii}(\xi)} d\xi\right\}$$

where:

$$p_i(0) = \left\{a_{ii}(0)\int_{\mathbb{R}_+} \frac{\phi_i(x_i)}{a_{ii}(x_i)} dx_i\right\}^{-1} \quad ; \quad \phi_i(x_i) \doteq \exp\left\{2\int_0^{x_i} \frac{b_i(\xi)}{a_{ii}(\xi)} d\xi\right\}$$

The separability condition is, of course:

$$a_{ij}(x_i, x_j) = 0 \quad \forall i \neq j$$

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Some Explicit Examples (3)

Jump case. Assuming constant drift, constant diagonal elements in a and jump-size distributions $k_i(x_i, z_i) = k_i(z_i)$ (independent of x_i) $\forall i$, one can show in this case:

$$\mathcal{L}[p_i](v_i) = \frac{\frac{1}{2}a_{ii}p_i(0^+)}{\frac{1}{2}a_{ii}p_i(0^+) + \mathbb{E}\lambda_i \mathbb{E}k_i + \frac{1}{2}a_{ii}v_i + \frac{\mathbb{E}\lambda_i}{v_i}\{\mathcal{L}[k_i](v_i) - 1\}}$$

where $\mathcal{L}[\cdot](\cdot)$ denotes Laplace transform and $p_i(0^+) = -2\frac{R_{ii}}{a_{ii}(0)}(R^{-1}\gamma)_i$, with $\gamma_l = b_l + \mathbb{E}\lambda_l \mathbb{E}k_l$ and $\mathbb{E}k_l = \int_{\mathbb{R}_+} z_l k_l(z_l) dz_l$. Therefore, for a specific set of jump-size distributions one can find the marginal densities $p_i's$ by taking inverse Laplace transform to the equation above, and then the required structure (for a product-form) on the off-diagonal diffusion elements by substituting in those marginals in the product-form condition.

A General Model

We consider the following reflected diffusion with positive and negative jumps, constrained to lie in the positive orthant of $\mathbb{R}^n$:

$$\begin{aligned} X_t &= X_0 + \int_0^t b(\omega, s, X_{s-})ds + \int_0^t \gamma(\omega, s, X_{s-})dW_s \\ &+ \sum_{0<s\leq t} \Delta X_s + \int_0^t R(\omega, s, X_{s-})dZ_s \end{aligned}$$

where X is an $\mathbb{R}_+^n$-valued *càdlàg* semi-martingale satisfying $\sum_{0<s\leq t} |\Delta X_s| < \infty$ a.s. for each $t > 0$, and:

- $b = (b_i)_{i\in\{1,\ldots,n\}} : \Omega \times \mathbb{R}_+^{n+1} \to \mathbb{R}^n$ and $\gamma = (\gamma_{ij})_{i,j\in\{1,\ldots,n\}} : \Omega \times \mathbb{R}_+^{n+1} \to \mathbb{R}^{n\times n}$ are random fields with $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+^{n+1}) - \mathcal{B}(\mathbb{R})$ measurable coefficients, both $(\mathcal{F}_t)_{t\geq 0}$-adapted for each fixed $x \in \mathbb{R}_+^n$.

A General Model (2)

and:

- $R = (R_{ij})_{i,j \in \{1,\ldots,n\}} : \Omega \times \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}^{n \times n}$ is a random field with $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+^{n+1}) - \mathcal{B}(\mathbb{R})$ measurable coefficients, $(\mathcal{F}_t)_{t \geq 0}$-adapted for each fixed $x \in \mathbb{R}_+^n$. Note the definition of R is only relevant for $(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \partial_0$: since X is *càdlàg* and Z is continuous, we have $\int_0^{\cdot} R(s, X_{s-}) dZ_s = \int_0^{\cdot} R(s, X_s) dZ_s$, and, moreover, each Z^i can only increase at times s when $X_s^i = 0$ (i.e., when $X_s \in \partial_i$), by definition of Z.
- In addition, we assume that $\mathbb{P}\left\{\omega \in \Omega : R(\omega, t, x) \in \mathcal{R}_n, \forall (t, x) \in \mathbb{R}_+ \times \partial_0\right\} = 1$, where $\mathcal{R}_n$ denotes the collection of all $n \times n$ completely-S matrices with the additional property that each of their principal submatrices is non-singular.

Note: State dependence occurs in a queueing context wthen routing matrix $Q(x) = P^T(x)$ depends on the buffer occupancy and $R(x) = (I - Q(x)$.

Main Results

- If $\exists i \in K \subseteq \{1,\ldots,n\}$ such that $\mathbb{P}\left\{\omega \in \Omega : a_{ii}(\omega,s,x) > 0, \forall x \in \cap_{k\in K}\partial_k\right\} = 1$ for a.e. $s \in [0,t]$, then:

$$a.s.: \quad m\{s \in [0,t] : X_s^j = 0, \forall j \in K\} = 0$$

  and:

$$a.e. \;\; s \in [0,t]: \quad P_s^X\{x \in \mathbb{R}_+^n : x_k = 0, \forall k \in K\} = 0$$

- If $\exists i,j \in K \subseteq \{1,\ldots,n\}, i \neq j$, such that $\mathbb{P}\left\{\omega \in \Omega : a_{ii}(\omega,s,x) > 0, \forall x \in \cap_{k\in K\setminus\{j\}}\partial_k\right\} = 1$ and $\mathbb{P}\left\{\omega \in \Omega : a_{jj}(\omega,s,x) > 0, \forall x \in \cap_{k\in K\setminus\{i\}}\partial_k\right\} = 1$ for a.e. $s \in [0,t]$, then $\forall q \in K$ we have a.s.:

$$\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\} dZ_s^q = 0$$

Main Results (2)

- If for each $i \in \{1, \ldots, n\}$ we have $\mathbb{P}\{\omega \in \Omega : a_{ii}(\omega, t, x) > 0, \forall x \in \partial_i\} = 1$ for a.e. $t \in \mathbb{R}_+$, then for each $i \in \{1, \ldots, n\}$ we have a.s.:

$$Z_\cdot^i = \frac{1}{2} \int_0^\cdot \frac{L^i(ds, 0)}{R_{ii}(s, X_s)}$$

- **Remark.** The previous results extend the boundary characterization already given for the case of non-random drift and diffusion coefficients, and constant reflection matrix.

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Case of a Wedge as State Space

Consider:

$$Y_t = Y_0 + \int_0^t b(\omega, s, Y_{s-})ds + \int_0^t \gamma(\omega, s, Y_{s-})dW_s$$
$$+ \sum_{0<s\leq t} \Delta Y_s + \int_0^t R(\omega, s, Y_{s-})dZ_s$$

where now:

- The state space of Y is $\mathcal{S}_Y \subseteq \mathcal{S} \doteq \mathbb{R}^n_+$, $\mathcal{S}_Y \doteq (N^T)^{-1}\mathcal{S} = \{y \in \mathcal{S} : N^T y \geq 0\}$, with the columns of the non-singular matrix $N \in \mathbb{R}^{n\times n}$ corresponding to the inner-normals to the boundary faces of $\mathcal{S}_Y$.
- Each Z^i satisfies the condition $\int_{\mathbb{R}_+} \mathbf{1}\{Y_s \notin \partial_i^Y\}dZ_s^i = 0$.

Case of a Wedge as State Space (2)

and:

- The random field R satisfies the corresponding generalized condition $\mathbb{P}\left\{\omega \in \Omega : (< \hat{d}_i, R_j(\omega, t, y) >)_{i,j \in \{1,\ldots,n\}} \in \mathcal{R}_n, \forall (t, y) \in \mathbb{R}_+ \times \partial_0^{\Upsilon}\right\} = 1$, where $R_j = (R_{ij})_{i \in \{1,\ldots,n\}}$, i.e., the j-th column of R for each $(\omega, t, y) \in \Omega \times \mathbb{R}_+ \times \mathcal{S}_{\Upsilon}$.

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Main Result

Assume for each $i \in \{1, \ldots, n\}$ we have $\mathbb{P}\left\{\omega \in \Omega : \|\gamma^T(\omega, t, y)\hat{d}_i\|_2 > 0, \forall y \in \partial_i^Y\right\} = 1$ for a.e. $t \in \mathbb{R}_+$. Let $\emptyset \neq K \subseteq \{1, \ldots, n\}$. Then, we have:

- $m\left\{t \in \mathbb{R}_+ : Y_t \in \cap_{k \in K} \partial_k^Y\right\} = 0$ a.s.
- $P_t^Y\left\{\cap_{k \in K} \partial_k^Y\right\} = 0$ for a.e. $t \in \mathbb{R}_+$.
- If $|K| \geq 2$, then $\int_{\mathbb{R}_+} \mathbf{1}\{Y_s \in \cap_{k \in K} \partial_k^Y\} dZ_s^q = 0$ a.s. for each $q \in K$.
- $Z_\cdot^i = \frac{1}{2} \int_0^\cdot \frac{L_Y^i(ds)}{<\hat{d}_i, R_i(s, Y_s)>}$ a.s. for each $i \in \{1, \ldots, n\}$, where $(L_Y^i(t))_{t \geq 0}$ is the local time at level zero for $X^i \doteq (N^T Y)^i$, i.e.:

$$L_Y^i(t) = L_X^i(t, 0) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}\{0 \leq X_s^i \leq \epsilon\} \|\gamma^T(s, N^{-T} X_s)\hat{d}_i\|_2^2 ds$$

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Product-form Stat. Distributions

Consider first:

$$dX_t = b(X_{t-})dt + \gamma(X_{t-})dW_t + \Delta X_t + R(X_{t-})dZ_t$$

where everything is as in the case with constant R, but now $R = R(x)$, $x \in \mathbb{R}^n_+$. Assume $R_{ij}(x) = R_{ij}(x_i)$. Note since $R_j(x) = (R_{1,j}(x_1), \ldots, R_{j,j}(0), \ldots, R_{n,j}(x_n))$ only comes into play upon hitting the j-th face, then this is equivalent to ask for $R_{ij}(x) = R_{ij}(x_i, x_j)$. Then, the stationary density p is in product-form if and only if for each $i, j \in \{1, \ldots, n\}$, $i \neq j$, we have:

$$a_{ij}(x_i, x_j) = \frac{R_{ji}(x_j)}{2R_{ii}(0)} \frac{a_{ii}(0+)p_i(0+)}{p_i(x_i)} \int_{x_i}^{\infty} p_i(\xi)d\xi \quad + \quad (i \leftrightarrow j)$$

for P_{ij}-a.e. $(x_i, x_j) \in \mathbb{R}^2_+$, where P_{ij} denotes the joint (stationary) law of the tuple (X^i_t, X^j_t) in $\mathbb{R}^2_+$.

Product-form Stat. Distributions (2)

Some examples:

- **Continuous Case.** Assume $\lambda_i \equiv 0$ (i.e., there are no jumps), $a_{ii}(x_i) > 0$ for all $x_i \in \mathbb{R}_+$, and smooth coefficients. Then:

$$p_i(x_i) = \frac{a_{ii}(0)p_i(0)}{a_{ii}(x_i)} \exp\left\{ \int_0^{x_i} \frac{2b_i(\varsigma) + q_i(\varsigma)}{a_{ii}(\varsigma)} d\varsigma \right\}$$

where:

$$q_i(x_i) \doteq \sum_{j \neq i} \frac{a_{jj}(0)p_j(0)}{R_{jj}(0)} R_{ij}(x_i)$$

>From the normalization condition, for each i:

$$1 = a_{ii}(0)p_i(0) \int_0^{\infty} \frac{1}{a_{ii}(x_i)} \exp\left\{ \int_0^{x_i} \frac{2b_i(\varsigma) + q_i(\varsigma)}{a_{ii}(\varsigma)} d\varsigma \right\} dx_i$$

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Product-form Stat. Distributions (3)

Some examples:

- *Càdlàg* **Case.** Assume $b_i(x_i) = b_i$ (a constant), $a_{ii}(x_i) = a_{ii} > 0$ (an strictly positive constant) and $K_i(x_i, dz_i) = k_i(x_i, z_i)dz_i = k_i(z_i)dz_i$ with $k_i(z_i) \equiv 0$ for $z_i < 0$. Then:

$$
\begin{aligned}
0 &= \mathcal{L}[p_i](\alpha_i)\left\{\frac{1}{2}\alpha_i a_{ii} - b_i + \frac{\overline{\lambda}_i\mathcal{L}[k_i](\alpha_i) - \overline{\lambda}_i}{\alpha_i}\right\} \\
&- \frac{1}{2}a_{ii}p_i(0+) - \frac{1}{2}\sum_{j \neq i}\frac{a_{jj}p_j(0+)}{R_{jj}(0)}\mathcal{L}[R_{ij}p_i](\alpha_i)
\end{aligned}
$$

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Product-form: Random Case

Consider now:

$$dX_t = b(\omega, X_{t-})dt + \gamma(\omega, X_{t-})dW_t + \Delta X_t + R(\omega, X_{t-})dZ_t$$

Set:

- $\hat{b}_i(X_0(\omega)) = \mathbb{E}[b_i(X_0)|\sigma(X_0)](\omega)$ a.s.
- $\hat{a}_{ij}(X_0(\omega)) = \mathbb{E}[a_{ij}(X_0)|\sigma(X_0)](\omega)$ a.s.
- $h_{ij}(X_0(\omega)) = \mathbb{E}\left[\frac{R_{ij}\left(X_0^{0_j}\right)}{R_{jj}\left(X_0^{0_j}\right)} a_{jj}(X_0) \middle| \sigma(X_0)\right](\omega)$ a.s. $(i \neq j)$.

Assumptions: $\hat{b}_i(x) = \hat{b}_i(x_i)$, $\hat{a}_{ij}(x) = \hat{a}_{ij}(x_i, x_j)$ and $h_{ij}(x) = h_{ij}(x_i, x_j)$.
Moreover, $\hat{a}_{ii}(0+)$ and $h_{ij}(x_i, 0+)$ exist and are finite.

Product-form: Random Case (2)

- **Main Result.** Then, the stationary density p is in product-form if and only if for each $i, j \in \{1, \ldots, n\}, i \neq j$, we have:

$$\hat{a}_{ij}(x_i, x_j) = \frac{h_{ji}(x_j, 0+)p_i(0+)}{2p_i(x_i)} \int_{x_i}^{\infty} p_i(\xi)d\xi \quad + \quad (i \leftrightarrow j)$$

for P_{ij}-a.e. $(x_i, x_j) \in \mathbb{R}_+^2$.

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Existence Results

Consider:

$$dX_t = b(X_{t-})dt + \gamma(X_{t-})dW_t + \Delta X_t + R(X_{t-})dZ_t \qquad (*)$$

with $b_i(x) = b_i(x_i)$, $a_{ij}(x) = a_{ij}(x_i, x_j)$, $R_{ij}(x) = R_{ij}(x_i)$, and the following jump structure for each component X^i:

$$\sum_{0<s\leq\cdot} \Delta X^i_s(\omega) = \int_0^{\cdot} \int_{E_i} c_i(X^i_{s-}(\omega), r_i) q_i(\omega, ds, dr_i)$$

- q_i is an $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measure on $\mathbb{R}_+ \times E_i$ with intensity measure $\hat{q}_i(ds, dr_i) = \lambda_i G_i(dr_i)ds$, λ_i is the jump intensity, G_i is a probability measure on $(E_i, \mathcal{B}(E_i))$ with E_i an arbitrary Polish space (e.g., $E_i = \mathbb{R}$), and $\{q_i(\omega, ds, dr_i)\}_{i=1}^n$ are independent.
- Each c_i is such that $c_i(y, r_i) > -y$ for all $y \in \mathbb{R}_+$ and $r_i \in E_i$.

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Existence Results (2)

- Assume $R_{ij}(\cdot) \leq 0 \; \forall i, j \in \{1, \ldots, n\}, i \neq j$, and that, with $Q(\cdot) \doteq I - R(\cdot)$, there exists $\overline{V} \geq 0$ (elementwise) such that $Q(x) \leq \overline{V}$ (elementwise) $\forall x \in \mathbb{R}^n_+$ and the spectral radius of $\overline{V}$ is < 1. (Without loss of generality we take $R_{ii}(\cdot) \equiv 1$.)
- Under the usual Lipschitz and linear growth conditions on b, γ, R and the jump amplitudes, equation (*) possesses a (pathwise) unique strong solution for each initial condition $X_0 = x \in \mathbb{R}^n_+$ and, denoting as X^x the corresponding solution starting from x, the family $\{X^x\}_{x \in \mathbb{R}^n_+}$ is Feller-Markov.
- Assume also the following irreducibility condition. For each $t > 0$, $x \in \mathbb{R}^n_+$ and $A \in \mathcal{B}(\mathbb{R}^n_+)$, $P_x(t, A) = 0$ if and only if $m(A) = 0$, where $P_x(t, \cdot)$ denotes the law of X_t in $\mathbb{R}^n_+$ and $m(\cdot)$ Lebesgue measure in $\mathbb{R}^n_+$. (Note this condition is satisfied when for example the diffusion matrix is uniformly elliptic.)

Existence Results (3)

- Also, among some other technical assumptions, assume $\delta \doteq \min_{i\in\{1,\ldots,n\}} \inf_{y\in\mathbb{R}_+} a_{ii}(y) > 0$.

Key Stability Condition. Set $\psi_i(y) \doteq b_i(y) + \lambda_i \int_{E_i} c_i(y, r_i) G(dr_i)$ for each $i \in \{1, \ldots, n\}$ and $y \in \mathbb{R}_+$, and $\psi(x) \doteq (\psi_1(x_1), \ldots, \psi_n(x_n))$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$. Then, with $Q(\cdot)$ as before, there exists $\underline{V} \geq 0$ (elementwise) such that $Q(x) \geq \underline{V}$ (elementwise) for all $x \in \mathbb{R}^n_+$ and:

$$\Theta \doteq \max_{i\in\{1,\ldots,n\}} \sup_{x\in\mathbb{R}^n_+} \left[(I - \underline{V})^{-1} \psi(x) \right]_i < 0$$

Existence Results (4)

Definition 1. Fix an $\epsilon > 0$. For each $i \in \{1, \ldots, n\}$, let T_0^i be the first hitting time of 0 for X^i after ϵ, i.e., $T_0^i \doteq \inf\{t \geq \epsilon : X_t^i = 0\}$.

Under the key stability assumption in the previous slide, it is easy to see that $\mathbb{E}_0[T_0^i] < \infty$ for each $i \in \{1, \ldots, n\}$, where $\mathbb{E}_0[\cdot]$ denotes expectation under $\mathbb{P}_0$, the law of X when starting from 0.

Definition 2. For each $i \in \{1, \ldots, n\}$, define the function $\eta_i : \mathbb{R}_+ \to \mathbb{R}_+$ by setting for $y \in \mathbb{R}_+$:

$$\eta_i(y) \doteq \left(\mathbb{E}_0\left[T_0^i\right]\right)^{-1} \frac{\mathbb{E}_0\left[L_i(T_0^i, y)\right]}{a_{ii}(y)}$$

It can be shown that each $\eta_i(\cdot)$ is strictly positive, bounded and continuous over $\mathbb{R}_+$, and such that $\int_0^\infty \eta_i(y)dy = 1$.

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Existence Results (5)

Main Result. Define the probability measure Π on $(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n))$ by setting:

$$\Pi(A) = \int_A \eta_1 \cdots \eta_n dm = \int_A \eta_1(x_1) \cdots \eta_n(x_n) dx_1 \cdots dx_n$$

for each $A \in \mathcal{B}(\mathbb{R}_+^n)$. Assume for each $i, j \in \{1, \ldots, n\}, i \neq j$, we have:

$$a_{ij}(x_i, x_j) = \frac{a_{ii}(0)\eta_i(0)}{2} \frac{R_{j,i}(x_j)}{\eta_i(x_i)} \int_{x_i}^{\infty} \eta_i(\xi) d\xi \quad + \quad (i \leftrightarrow j)$$

for all $x_i, x_j \geq 0$.

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Existence Results (6)

Main Result (cont.). Then, Π is the unique stationary (or invariant) distribution for X, in that it is the unique probability measure on $(\mathbb{R}^n_+, \mathcal{B}(\mathbb{R}^n_+))$ such that, for each $t \geq 0$ and bounded Borel measurable function $f : \mathbb{R}^n_+ \to \mathbb{R}$:

$$\int_{\mathbb{R}^n_+} \mathbb{E}_x \left[f(X_t) \right] \Pi(dx) = \int_{\mathbb{R}^n_+} f(x) \Pi(dx)$$

Moreover, the product-form Radon-Nikodym derivative:

$$\frac{d\Pi}{dm}(x) = \eta_1(x_1) \cdots \eta_n(x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$$

is strictly positive and belongs to the class $\mathcal{C}_b(\mathbb{R}^n_+)$.

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Existence Results (7)

Main Result (cont.). In addition, for each initial distribution μ on $(\mathbb{R}^n_+, \mathcal{B}(\mathbb{R}^n_+))$ we have:

$$\lim_{t\uparrow\infty} \int_{\mathbb{R}^n_+} P_x(t, A)\mu(dx) = \Pi(A)$$

for each $A \in \mathcal{B}(\mathbb{R}^n_+)$. In particular, for each $f \in \mathcal{C}_b(\mathbb{R}^n_+)$ we have:

$$\lim_{t\uparrow\infty} \int_{\mathbb{R}^n_+} \mathbb{E}_x \left[f(X_t) \right] \mu(dx) = \int_{\mathbb{R}^n_+} f(x)\Pi(dx)$$

that is, the measure $\int_{\mathbb{R}^n_+} P_x(t, \cdot)\mu(dx)$ converges weakly to $\Pi(\cdot)$ as t goes to infinity.

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Existence Results (8)

Key Ideas for the Proof.

- Use the characterization:

$$Z^i_\cdot = \frac{1}{2}\int_0^\cdot \frac{L_i(ds,0)}{R_{ii}(X_s)} = \frac{1}{2}L_i(\cdot,0)$$

- Then use the time occupation density interpretation of semi-martingale local times, i.e., that $\forall f : \mathbb{R} \to \mathbb{R}$, bounded Borel measurable function, we have a.s.:

$$\int_0^t f(X^i_{s-})a_{ii}(X^i_s)ds = \int_{-\infty}^{+\infty} f(r)L_i(t,r)dr$$

for each $i \in \{1, \ldots, n\}$.

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The general case

In the case of constant R sufficient conditions for the existence of and convergence to the stationary distribution were geven by Budhiraja and Atar (EJP2003)

Key Stability Condition. Set $\psi_i(y) \doteq b_i(y) + \lambda_i \int_{E_i} c_i(y, r_i) G(dr_i)$ for each $i \in \{1, \ldots, n\}$ and $y \in \mathbb{R}_+$, and $\psi(x) \doteq (\psi_1(x_1), \ldots, \psi_n(x_n))$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$. Let $\sigma(Q) < 1$ and $I - Q > 0$

$$\Theta \doteq \max_{i \in \{1,\ldots,n\}} \sup_{x \in \mathbb{R}^n_+} \left[(I - \underline{Q})^{-1} \psi(x) \right]_i < 0$$

Concluding Remarks

- An appropriate boundary behavior characterization for reflected jump-diffusions is key to the study of product-form stationary distributions.
- No sufficient conditions are available in the general case with state-dependent reflections- reflection map is not Lipschitz. However we conjecture that our key condition is sufficient for the existence.

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Concluding remarks

Why are we interested: in general, no results are available to show the existence of stationary distributions for networks with ergodic arrivals- except in the monotone separable case (see Foss and Baccelli).

One way: establish fluid model is stable and show the diffusion scaling limit possesses an invariant distribution. For this we need tightness and an interchange of limits argument to go through. This will avoid the construction of a Lyapunov function (indeed it is hard to identify).

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